A Model for Scaling in Firms' Size and Growth Rate Distribution

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Abstract

We introduce a simple agent-based model which allows us to analyze three stylized facts: a fat-tailed size distribution of companies, a 'tent-shaped' growth rate distribution, the scaling relation of the growth rate variance with firm size, and the causality between them. This is achieved under the simple hypothesis that firms compete for a scarce quantity (either aggregate demand or workforce) which is allocated probabilistically. The model allows us to relate size and growth rate distributions. We compare the results of our model to simulations with other scaling relationships, and to similar models and relate it to existing theory.

1 Introduction

Growth processes have been the object of active research since the ground-laying work of Gibrat [9] who described growth as a multiplicative stochastic process. By assuming that growth rates are random variables, and studying the time evolution of the system, he obtained a lognormal distribution of company sizes, i.e. a heavy-tailed distribution. The topic has since then received wide attention [4]. Recent empirical studies suggest that the firm size distribution follows a Zipf (power law) distribution [15, 28]. Various different models exist for the formation of such distributions with power-law tails. In the context of firm growth, the well-known model by Simon [36] explains a power law for firm size distribution on the basis of a process introduced by Yule [35]. In this model, the system is constantly growing, not only in firm size, but also in the number of firms. The exponent of the power law depends on the frequency of new firms, and approaches 1 if this frequency is low. However the need that the system has to grow continuously for the formation of a power law has led to criticism [35, 39]. For systems of constant global size, which are the focus of this paper, models exist that explain the formation of fat tails by multiplicative stochastic processes. For such systems with multiplicative noise, no result of the generality of the central limit theorem exists [37]. However, it has been shown that the stationary distribution has a power law tail in the lowest order approximation, when the noise satisfies some stringent conditions [10, 14, 11, 27].

More recently, Stanley et al [19] uncovered two empirical features that an accurate theory on firm growth should explain: the growth rate frequencies exhibit an exponential (Laplacian) decay – giving rise to a tent shape in logarithmic scale –, and the variance of the growth rate scales with the company size n as $\sigma(n) \propto n^{-\beta}$. This means that the simple assumptions by Gibrat and others, who assume multiplicative noise to be independent of firm size, are at odds with the data. The empirically

determined values of β (typically ≈ 0.2) depend on the studied system. Many papers have provided further evidence of these two findings in many growth processes: firm growth [19] ($\beta = 0.15$), [38] ($\beta = 0.18$), [30] ($\beta = 0.28$), [21] ($\beta = 0.3$) (the latter authors also consider bird populations and mutual funds), a country's GDP growth [29] ($\beta = 0.15$), citations in scientific journals [31] ($\beta = 0.22$) and the growth rate of crime [32] ($\beta = 0.36$). The multitude of examples suggest that the process generating both a tent-shaped growth rate distribution and a scaling exponent $\beta \neq 0$ is simple and universal.

A number of sophisticated models giving rise to a tent shaped growth rate distribution have been proposed, following different approaches. Botazzi and Secchi [3] predict a tent-shaped (Laplacian) growth rate distribution as being the result of a number of abstract shocks drawn from a Polya urn, without adressing the question of the variance scaling exponent. In order to obtain the variance scaling, many models assume that firms have an internal structure, i.e. they are composed of subunits [22, 30, 26] In Schwarzkopf et al.[21], the probability that firms' subunits reproduce themselves follows a power law. As a result, the aggregated growth rate distribution is also a power law: it is not a collective phenomenon but holds already at the individual level.

Another interesting attempt [31] assumes that the growth rate variance depends on the elements' size (which are citations in their case), and numerically obtains a fat-tailed size distribution. The tent-shaped growth rate distribution is however not a result of the model but a hypothesis on individual level.

Most models explaining the tent-shaped growth rate distribution and the variance scaling relation do not attempt to explain simultaneously the formation of the fat-tailed firm size distribution. This task is tricky: Existing models for power law tails via multiplicative noise assume the growth rate to be independent of the firms' size, which seems in conflict with a scaling exponent $\beta \neq 0$. However, for both a power law firm size distribution and a scaling exponent $\beta > 0$ there is empirical evidence.

We address this issue in this paper, both theoretically and numerically with a simple agent-based model comprising firms and employees. We distiguish between collective phenomena at the firm level and at the level of the macroeconomy. In contrast to the models cited above, we investigate the hypothesis that the same process accounts for the tent-shaped growth rate distribution, its variance scaling relation and the fat-tailed size distribution of firms.

The paper is organized as follows. In section 2 we introduce the model. In section 3 we discuss the size distribution with heavy tails obtained with our model, and give an intuitive explanation in terms of a Langevin equation, which is our first main result. In section 4, we present the growth rate distribution of the model, which is our second main result, and discuss some consequences of binning the data. In section 5 we compare empirically found scaling exponents to our model and point out possible extensions of it. In section 6, we discuss a theoretical aspect of the model and compare it to further literature, then we conclude and point out extensions.

2 The model

Our agent-based model comprises companies and workers, N_c and N_w, which are both constant numbers. At each iteration, firms hire workers in order to produce goods, which are then sold. Based on their profits, firms adjust the quantity to produce in the next time step. They compete for a limited resource, which can be purchasing power of customers, or workforce. Firms demand a discrete quantity of this resource, proportional to their size. The proportionality constant reflecting the profits is $(1+\mu)$, with $\mu > 0$, which in this paper is assumed to be the same for all firms.[1] Since $\mu > 0$, firms attempt to increase their size, and the stationary state of the system corresponds to full employment.

A matching algorithm guarantees that the resource is attributed with the same probability to every demand: every open position has the same probability of being filled by a worker. This hypothesis, which is equivalent to the microcanonical ensemble in statistical physics, and is also used elsewhere in the context of growth processes [24, 22]. Interesting dynamics arise if there is shortage of the resource: since the $N_w\,(1+\mu)$ positions are covered at random, the actual number $n_{i,t+1}$ of employees hired by a firm i at time $t+1$, may be smaller than their joboffer $\hat{n}_{i,t} = n_{i,t}(1+\mu)$. It can even be smaller than the number of employees $n_{i,t}$ in the preceding period, which confronts us with the difficulty that firms may receive no worker at all and vanish. The number of active firms would decrease continuously, and workers would eventually accumulate in a monopoly, which is avoided in our model by the introduction of new firms. To maintain N_c constant, extinct firms are replaced with by new ones[2] , initialized with a number of workers $n_{i,t}^{new}$ drawn from a distribution $\mathcal{F}(n^{new})$. New firms contribute to the total demand for workforce in the next period with the quantity $(1+\mu)n_{i,t}^{new}$. Workers do not stay at their firm but are newly placed at every iteration. However our results would hold if only a (constant) fraction of workers of each company were newly placed.

An analytical formulation of our model with a strictly constant number of workers is complicated. We relax this constraint, and assume that it is satisfied on average: $\langle \sum_i n_{i,t} \rangle = N_w$, to obtain equations that allow to understand the evolution of the system. Then, the probability for an open position to be filled can be written as

$$p = \frac{N_w}{\sum_i \hat{n}_i} \,, \tag{1}$$

where $\sum_i \hat{n}_i = N_w + \sum_i n_i^{new}$. Since $\sum_i n_i^{new} \ll N_w$ we may approximate $p \approx (1+\mu)^{-1}$. Under such conditions the probability that a firm with n_i workers will receive k_i workers in the next period is the binomial distribution

$$P(k_i|n_i) = \binom{\hat{n}_i}{k_i} p^{k_i} (1-p)^{\hat{n}_i - k_i} \,, \tag{2}$$

with mean $\hat{n}_i\, p = n_i$, which is precisely the number of workers of firm i at the previous time step, and variance $\hat{n}_i\, p\,(1-p) = n_i \mu/(1+\mu)^2$. The probability that a firm does not get any worker, in which case it disappears, is $P(0|n_i) = (1-p)^{\hat{n}_i}$.

For large n_i, the binomial distributions may be approximated by Gaussian distributions, exhibiting the same n-dependence of their variance. Another implementation uses a different rounding method by which firms determine their job offer, such that the Gaussian distribution is also a good approximation for small firms. It is used in simulations and can be seen as growth of entirely independent subunits,

[1] A scenario with heterogeneous μ_i of firms is published elsewhere [17]. In that setting, the limited resource is purchasing power of workers, whereas in this paper it is workforce.

[2] It is also possible to re-insert firms at a constant rate, in which case a level of active firms will become stationary after some time. For this paper, we used a strictly constant number of active firms, since this allows us to present results of systems of identical size.

where subunits are jobs (see appendix A). The Gaussian approximation of the probability for a firm of size $n_{i,t}$ to reach size $n_{i,t+1}$ is written as

$$\mathcal{P}(n_{i,t+1} = k_i | n_{i,t}) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{k_i - n_{i,t}}{\sigma_i}\right)^2} . \tag{3}$$

If the growth rate of a firm is defined as

$$g_{i,t} = \frac{n_{i,t+1}}{n_{i,t}} , \tag{4}$$

equation (3) yields for the growth rate probability density $\mathcal{G}$ (dropping the index t):

$$\mathcal{G}(g_i | n_i) = \sqrt{\frac{n_i}{2\pi\, c}} e^{-\frac{1}{2}\frac{n_i}{c}\,(g_i - 1)^2} , \tag{5}$$

where $c = \mu/(1+\mu)^2$ for the binomial approximation[3]. The scaling exponent β of the growth rate's standard deviation, which is defined through

$$\sigma(n) \propto n^{-\beta} , \tag{6}$$

has the value $\beta = 0.5$. This value for β is a general feature of models that explain firm growth as being the sum of the growth of independent subunits. In the literature, subunits often represent the sectors in which the firm is active [38, 24, 22]; in our model these are jobs. We further discuss other values for β and the corresponding empirical evidence in section 5.

3 Analysis of the fat-tailed size distribution

Simulations of our model lead to a stationary state that exhibits a fat-tailed size distribution of firms shown in figure 1 (a), which is however not a power-law. In order to describe it, we briefly summarize the established theory on the formation of power laws, which we illustrate with numerical simulations, before describing the results of our model in that context.

3.1 Formation of power laws by multiplicative noise

We recall that the evolution of a random variable n is usually described by the following Langevin equation

$$n_{t+1} = g_t\, n_t + f_t, \tag{7}$$

with g_t a random multiplicative noise and f_t a random additive noise. Equation (7) has various applications in statistical physics, and is used here to determine the probability density fuction $\rho(n)$ in the stationary state. It has been shown in several ways that a multiplicative noise term g_t leads to the formation of distributions with power law tails for $\rho(n)$. For certain discrete noise distributions, it is even possible to derive $\rho(n)$ analytically [1, 12]. For more general noise distributions, $\rho(n)$ may be deduced under some general approximations [10, 18, 14, 39].

In the case of the size distribution of firms $\rho(n)$, a multilicative noise term g_t in equation (7) means that the size of each firm in the system changes by a multiplicative factor drawn from a distribution

[3]For the rounding method detailed in the appendix A, this constant is $\frac{2\mu}{(1+\mu)^2}$

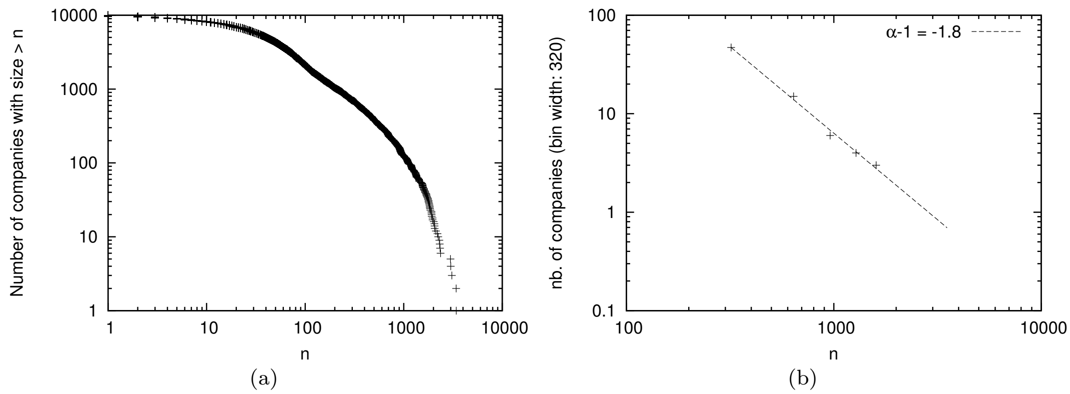


Figure 1: Snapshots (taken after 3900 iterations) of (a) the cumulative size distribution and (b) the frequency distribution of firm sizes, obtained from a simulation of our model with $N_w = 10^6$ workers and $N_c = 10^4$ companies. The size distribution is fat-tailed but its slope becomes steeper for larger firms. However this is not visible in (b) where firms sizes are binned. The slight irregularity at size 100 in (a) comes from the additive noise: in this setup newly started firms have a size distribution $\mathcal{F}(n)$ that is uniform between 1 and 100.

g, which is independent of the firm's current size. For instance, if g is drawn from a Gaussian for each firm, so would be the aggregated growth rate distribution of firms. This is illustrated with a simulation shown in figure (2), where firms' growth rates were drawn from such a Gaussian distribution (instead of the growth term from our model).

The intuitive interpretation for the necessity of the additive term f, already given in section 2, holds generally: for a noise distribution where g that can take values < 1, some firms have a nonzero probability of shrinking to size 0, i.e. they "die out". Since the number of companies should also be stationary in the stationary state, an additive noise f is needed [10], which means for instance to re-insert firms having shrunk to size zero, or to add some term to firms whose size is falling below a threshold, such as [40, 39], which is preventing firms from attaining size 0.

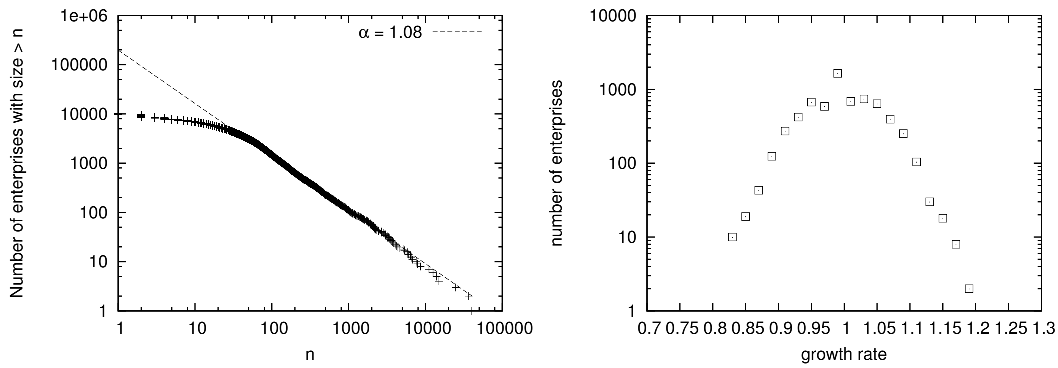
[10] showed that the size distribution can be derived by expanding its characteristic function in the stationary state. At the lowest order approximation, the cumulative distribution $P(n)$ exhibits a power law tail:

$$P(n' \geq n) \propto n^{-\alpha} \tag{8}$$

By re-inserting this expression into the condition for the stationary state, it is possible to derive [10, 39] the condition

$$\langle g^{\alpha} \rangle = 1\,. \tag{9}$$

which can be used to determine the exponent α if the distribution of g is known. We have numerically confirmed the exponents α calculated via equation (9) for Gaussian $G(g)$, varying both its standard deviation σ and the mean m around which the Gaussian is centered (see appendix B for technical details).



(a) (b)

Figure 2: Snapshots (taken after 3650 iterations) of the size (cumulative) (a) and growth rate (b) distributions, obtained from of a simulation with a (size-independent) Gaussian multiplicative noise, of a system with $N_w = 10^6$ workers and $N_c = 10^4$ companies. The Gaussian distribution appears parabolic in logscale.

3.2 Firm size distribution in our model

The question now is what changes in this result when the variance of the multiplicative noise term g_t is size-dependent. Numerical results of our model, which has a size dependent standard deviation of $\mathcal{G}(g|n)$, are shown in figure 1. The stationary state of the firms' size distribution clearly has a fat tail, which nonetheless decays faster than a power law.

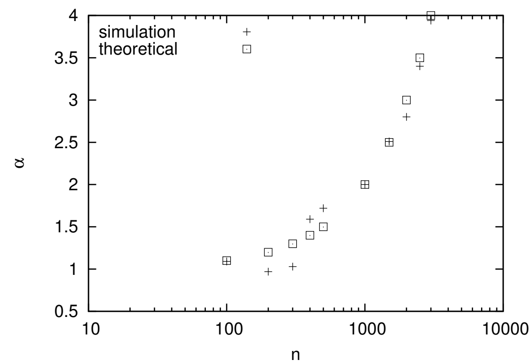
We have not succeeded in solving the master equation analytically for our model. However, the difference to the case where g is size-independent can be understood through a simple reasoning based on relation (9): assuming the Gaussian distribution for g, centered around a value m which is slightly smaller than 1, (9) yields that the smaller the variance of the noise, the larger the exponent α of the power law.

Interpreting $\alpha(n)$ as the slope of the tangent at size n of $P(n)$ in log-log representation, we expect that in our model, which has $\beta = 1/2$, the calculated $\alpha(n)$ increases with n, which is indeed the case, as shown in figure 3. For large sizes where additive noise f_t does not modify the distribution, this explanation seems to hold approximately. It is based on the assumption that since $\mathcal{G}(g|n)$ is Gaussian, mostly firms of size n_{t-1} close to n will reach size n at time t. Therefore, locally the variance g is assumed to be constant such that (9) holds.

We have performed simulations for several additive noise distributions $\mathcal{F}(n^{new})$. The slope of the fat-tailed cumulative size distribution depends on the distribution of $\mathcal{G}(g|n)$, and it is sensitive to the additive term f: the total demand for jobs then varies depending on $\sum_i n_{i,t}^{new}$. Since we keep the number of available workers constant, this leads to a different probability p for a position to be filled, and therefore to a different constant c in the growth rate variance. It also modifies the mean m around which the growth rate is centered. These factors have been taken into account for the calculation of $\alpha(n)$ in figure 3 by equation (9). Obtained values are close to the slopes fitted at certain values of n to the cumulative size distribution from the corresponding simulation.

The result from our model contains a general implication, which is the first of our two main results: as soon as σ of the growth rate is size-dependent, the fat-tailed size distribution for firms will decay

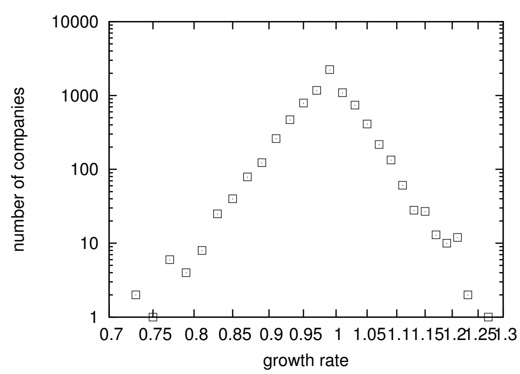
faster than power law, provided that the system has constant global size. The size-dependency of the growth rate variance might explain why many authors obtain very good fits of the size distribution by a lognormal distribution (see [9][37][3] and references therein), which also decays faster than power law. Simulations of another scaling exponent than in our model are presented in section 5, after our second main result in the next section.



(a)

Figure 3: For Gaussians g, $\langle g^{\alpha(n)} \rangle = 1$ yields $\alpha(n)$ as a function of firm size n (for scaling exponent β=0.5). Crosses represent the slopes of the tangent of the simulated cumulative size distribution at size n. After 3900 iterations, $N_c = 10^4$ and $N_w = 10^6$.

4 Growth rate probability distribution for specific firm size distributions



(a)

Figure 4: growth rate distribution of the same simulation as figure 1.

Figure 4 shows that despite the Gaussian $\mathcal{G}(g|n)$ in our model, in the stationary state $\mathcal{G}(g)$ exhibits a tent-shape. In fact, n-dependence makes that the growth rate distribution of $\mathcal{G}(g|n)$ is wider for

small firms and more narrow for big firms. Considering the aggregate $\mathcal{G}(g)$, small firms account for the fat tails of the tent-shaped $\mathcal{G}(g|n)$, and big firms for the peak. Theoretically, this result can be explained as follows: The aggregated growth rate distribution for the N_c firms can be written as

$$\mathcal{G}(g) = \frac{1}{N_c} \sum_{i=1}^{N_c} \mathcal{G}(g_i|n_i)\,, \tag{10}$$

or, in the continuous limit:

$$\mathcal{G}(g) = \int_0^\infty dn \mathcal{G}(g|n)\rho(n)\,, \tag{11}$$

where $\rho(n)$ is the firms' size distribution. Since we do not have an analytical expression for the simulated size distribution of our model, we evaluate the integral as an approximation for $\mathcal{G}(g|n)$ of our model and power-law size distributions of exponent α. For firms' size distributions $\rho(n) \propto n^{-\alpha-1}$ and scaling exponents $\beta = 0.5$ the integral yields

$$\mathcal{G}(g) = \int_0^\infty n^{0.5} \frac{1}{n^{\alpha+1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} n\,(g-1)^2} dn = \frac{1}{\sqrt{\pi}} 2^{-\alpha}\,(g-1)^{2\alpha-1} \Gamma\left(\frac{1}{2} - \alpha\right). \tag{12}$$

For $\alpha = 0$, it simplifies to $\mathcal{G}(g) = \frac{1}{|g-1|}$. In that case the slopes of $\mathcal{G}(g)$ are linear on a double-logarithmic scale, i.e. presenting a tent-shape. For $\alpha > 0.5$, the integral diverges at the origin and can only be integrated starting from a finite cutoff value n_0, since $n^{-\alpha-1}$ is not a normalized probability density. This includes the case of our model ($\alpha \approx 1$ and $\beta = 0.5$). If integrated from a cutoff n_0, equation (12) still yields a tent-shaped $\mathcal{G}(g)$, its width depending on the cutoff. Integral (12) can be generalized to values of β other than 0.5, which is interesting since empirical values are $\alpha \approx 1$ and $\beta \approx 0.25$. In that case, the condition for convergence of the integral becomes $\alpha > \beta$, and for $\alpha = 0$ the expression simplifies to $\mathcal{G}(g) = \frac{1}{2\,\beta\,|g-1|}$. The smaller is β, the less peaked $\mathcal{G}(g)$, which is intuitive, since if $\beta = 0$, the result is a Gaussian $\mathcal{G}(g)$.

Notice that the shape of $\mathcal{G}(g)$ is not very sensitive to the underlying size distribution: equation (12) yields an approximate tent-shaped $\mathcal{G}(g)$ even for exponential decay of $\rho(n)$. This suggests that despite the deviation from a power law which our model exhibits in its stationary state, the idea of performing integral (11) explains the observed tent-shape well.

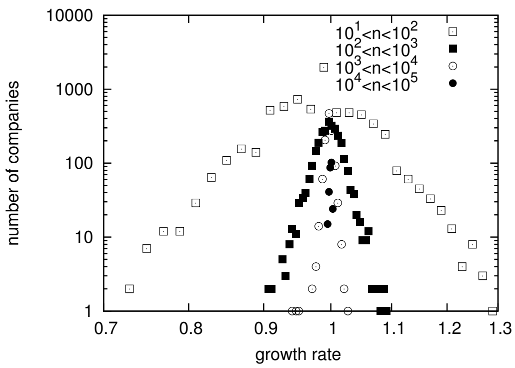
In the literature, the principle of performing this integral has been used in the model by [30] to obtain a tent-shaped growth rate distribution of a single firm. Other models [19, 29, 3, 31] do not perform the integral and do not clearly distinguish between the growth rate probability at firm level and at aggregate level, which they assume to follow a Laplacian distribution. The necessity to perform the integral in equation (11) is however independent of the assumed $\mathcal{G}(g|n)$. It turns out that the functional form of $\mathcal{G}(g)$ yields an approximate $1/|g-1|$ tent-shape for both Laplacian and Gaussian $\mathcal{G}(g|n)$. If it is integrated from a size cutoff, for low values of β, Laplacian $\mathcal{G}(g|n)$ yield a more peaked $\mathcal{G}(g)$. Since the growth rates have typically values close to 1, empirical evidence can often be fitted with a Laplacian (centered around 1) and a $1/|g-1|$- law equally well. However [30] find that the tails of the tent-shape exhibit power law decay rather than exponential decay of a Laplacian, substantiating our argument.

4.1 Artefacts from binning

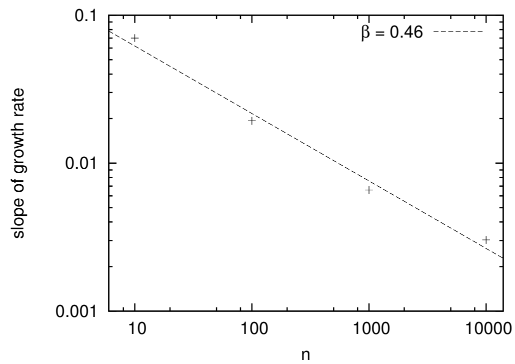
As already done for the size distribution, we discuss effects arising from binning data: The empirical data in [19, 29, 24, 31] show tent-shaped growth rate distributions of different widths depending on

firms size (or country's size or citations respectively), for which the authors propose a Laplacian fit. To do this, firms are grouped according to their size in large logarithmic bins. From the slopes of the growth rate distribution on logarithmic scale, $\sigma(n)$ and its scaling exponent β are determined.

Numerical simulations of our model show that aggregation of growth rates of firms within one order of magnitude is sufficient to obtain a growth rate distribution that resembles a tent-shape, in spite of the Gaussian $\mathcal{G}(g|n)$ in our model (see figure 5). The reason is again that $\rho(n) \approx 1/n^2$. This reveals that if an ensemble average is used to determine the shape of $\mathcal{G}(g|n)$, its functional form is only assessed correctly if the samples from the ensemble have precisely the same size. Already a size spectrum of one order of magnitude is enough to exhibit a different form for $\mathcal{G}(g|n)$. The value of β does not seem to depend on binning. Indeed we find again $\sigma \propto n^{-0.5}$.



(a)



(b)

Figure 5: Histograms for growth rate distributions. (a) Aggregate growth rate distribution for companies ranging from 1 to $5 \cdot 10^4$ employees from a system with 10^7 employees and a Zipf firm size distribution. (b) For Gaussian growth rate probability densities, the growth rate distribution of firms within one order of magnitude (its smallest size indicated in the plot) resemble a tent-shape, thus in qualitative agreement with the data shown in [19]. (c) The slopes of these approximate $1/|g-1|$-distributions follow the same scaling relation as the variance of the Gaussian growth rate probability densities $\sigma(n) \propto n^{-0.5}$.

The very simple underlying microscopic mechanisms, as well as compatibility with empirical evidence suggest that Gaussian functions might be a simpler alternative to the commonly assumed Laplacian shape for $\mathcal{G}(g|n)$. A further argument is that Gaussian distributions are conjugate priors to themselves, so they can be the result of the multiplication Gaussian growth rate probabilities for various reasons.

5 Extension and comparison to empirically found values of β

The empirical studies cited in the introduction find smaller values for β than our model. These are generally explained by firm-intern factors contributing to firm's growth, as by [19, 38, 22, 24]. Intuitively, if the growth of a company was entirely dependant on the decisions of its CEO, there would be no reason to assume that a company's size should affect its growth rate variance, so values of β between 0 and 0.5 are possibly due to a contribution of both internal hierarchical structure as well as

of a firm's size. An additional factor that can lower β are aggregate business fluctuations, which have not been considered in the derivation above: If the total available amount of the resource fluctuates, so do the means of the Gaussian growth rate distributions which are derived in equation (5). This is independent of the firm's size.

As a comparison to these empirical studies, we simulated a system with a Gaussian $\mathcal{G}(g|n)$ and a scaling exponent $\beta = 0.25$. This β is not the result of interactions in the job market as in our model, but firm's growth is simulated as Gaussian multiplicative noise where $\sigma \propto n^{-0.25}$, without specifying its microfoundations.

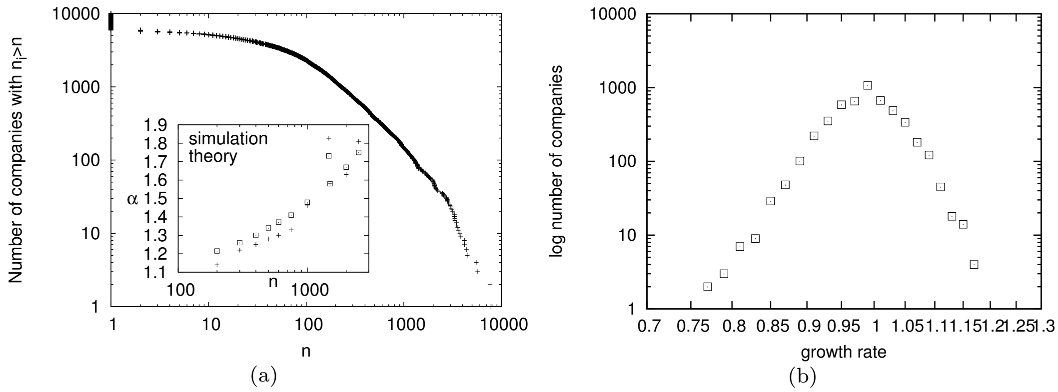


Figure 6: (a) cumulative size distribution and (b) growth rate distribution of a simulation with Gaussian multiplicative noise with a scaling relation of $\beta = 0.25$. (after 3000 iterations in a system with 10^6 workers and 10^4 companies)

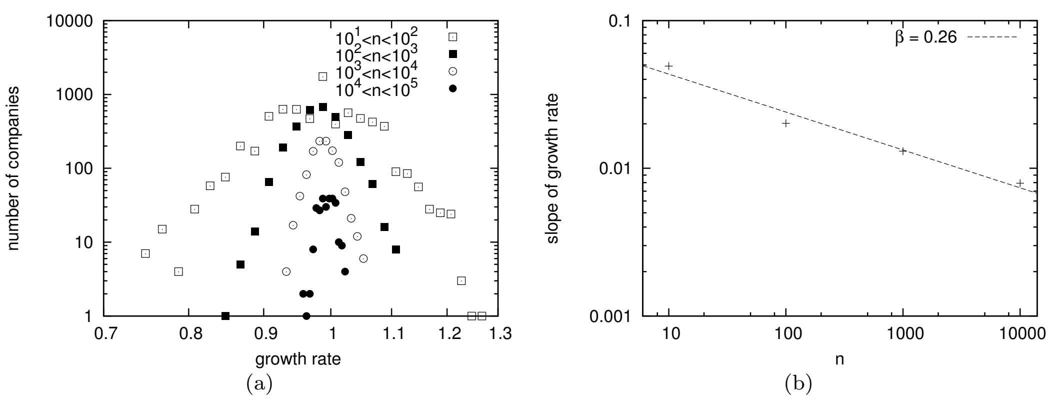


Figure 7: (a) Growth rate distribution where companies are clustered into size bins (system with 10^5 companies and 10^7 workers), (b) scaling exponent determined from the slopes in (a). It yields $\beta = 0.26$, in agreement with the scaling exponent $\sigma \propto n^{-0.25}$ of the Gaussian growth rate probability densities with which the system has been simulated.

The result of simulations shown in figures 6 and 7 are as expected in between pure multiplicative noise $\beta = 0$ (figure 2) and our model with scaling exponent $\beta = 0.5$ (figure 1 and 4). Although the cumulative size distribution on log-log-scale appears almost linear in the range $n \in [100; 1000]$ (figure 6), fitting slopes to it shows the n-variation of α, which is half as pronounced for $\beta = 0.25$ compared to $\beta = 0.5$. In contrast, the slopes of the tent-shaped growth rate distribution appear less linear than for $\beta = 0.5$. From these simulations and the previous discussions we conclude that there is a tradeoff if Gaussian $\mathcal{G}(g|n)$ are assumed: the more linear the size distribution is on double-logarithmic scale, the less peaked the growth rate distribution is. Whether this Gaussian hypothesis can indeed explain data depends on how well a power law fits the cumulative firm size distribution, and how peaked the tent-shaped growth rate distribution arising from the same dataset is. The strength of this simple hypothesis is that it would allow the explanation of heavy-tailed growth fluctuations as a collective phenomenon on aggregate level, without having to assume them on firm level, as e.g. [21] do. Even if a Laplacian $\mathcal{G}(g|n)$ is assumed, as many authors do, the problem remains that $\beta \neq 0$ will lead to a size-dependent approximate power law exponent $\alpha(n)$. Although for Laplacian $\mathcal{G}(g|n)$ the n-dependency of exponent α is half as pronounced as for Gaussians, it would be visible in data for for size distributions.

6 Discussion

Having presented the stationary state of our model (which has $\beta = 0.5$), as well as simulations of systems with $\beta = 0$ and $\beta = 0.25$ for comparison, we return to some theoretical aspects of our model. Its dynamics can be described on three levels: The noise on elementary (i.e. job) level is the same for every element, which can double, vanish, or stay constant (see appendix A). Iterated more than once, this process differs from multiplicative noise[4]: as soon as a job doubles, in the next iteration both jobs follow the dynamics separately. This elementary level allows for the calculation of the evolution of the size of the companies, which are the second level. On that level, the growth rate probability density is Gaussian, with a size-dependent variance.

Due to this dependency integral (11) becomes non trivial. The tent-shaped growth rate distribution of companies only holds at the level of a whole economy, which is the third level.

An analogy can be drawn to a physical system, where, due to long-range interactions, the statistics of an element and the statistics of the ensemble can diverge. [25] describe this by the term superstatistics, i.e. statistics of statistics, stating that in physical systems with fluctuations, the Boltzmann factor of the system is obtained by integrating the Boltzmann factors of every subsystem over their inverse temperatures. The analogy to the model presented here is the following: Instead of a Boltzmann factor, the quantity that governs the dynamics of a single firm is $\mathcal{G}(g|n)$, which is depending on n. This n-dependence is the result of long-range interactions: the hypothesis that every job is taken with the same probability implies that every open position interacts with every available employee. For Gibrat's model, performing an integral analogous to equation (11) is not necessary, since the interesting quantity is the growth rate, which is size-independent. For other systems with multiplicative noise, an integral analogous to (11) may not be omitted. Physical systems with multiplicative noise, where the dynamics depend on the square of a Gaussian variable exhibit Tsallis statistics [25][14]. Links of growth processes to nonextensive thermodynamics are presented in [33, 18] and for the case of the tent-shaped growth rate distribution [31]and [34].

Finally, we compare our model to two existing models which also exhibit $\beta = 0.5$. [1] presents a

[4]In contrast, the model by [26] use an elementary level which follows a dynamics with multiplicative noise. However, since the constituent elements therefore exhibit a power law size distribution, it is more difficult to interpret them.

micro-founded model for city growth, in which they study two scenarios, of which one corresponds to pure multiplicative noise ($\beta = 0$) and yields a power law, and one (which they term linear case) in which the growth rate variance has exponent $\beta = 0.5$. For the latter scenario, a size distribution other than a power law is derived. This setting differs from our model in that city-dwellers do not change the city all at the same time. It corresponds to our model if we simulated it in sequential update, a situation where workers drawn at random can change company, and the probability of joining a particular company is proportional to its size. In sequential update, $\mathcal{G}(g|n)$ would not be a distribution (as it is in our model as presented, which corresponds to parallel update) but just one value, proportional to the current size. We have simulated these sequential dynamics for comparison. If the statistics of $\mathcal{G}(g|n)$ are calculated after a given number of changes at the level of workers, very similar statistics to our tent-shaped $\mathcal{G}(g)$ are obtained.

The second interesting model with $\beta = 0.5$ is the widely used model by Simon[36], which has also binomial $\mathcal{G}(g|n)$ [39]. Simon's model exhibits a power law for firm size distribution, which is due to the fact that the system is constantly growing in employees and in the number of firms. However, it has been stated before that the power law is not been found if these two assumptions are not satisfied [41].

7 Conclusion

We have introduced a simple agent-based model where the growth rate of a firm is the result of the constraints it faces in the markets, in this case, the job market. Its growth rates of firms are size-dependent and exhibit the scaling exponent $\beta = 0.5$, as do other models that describe firm growth as the results of independent random processes. In order to keep its size constant, we need to introduce new firms to compensate the outgoing flow. The firm size distribution at the stationary state decays faster than power law, even though for its probability density in a binned representation it is not distinguishable from a power-law. On the basis of this model we show, by using theory on systems with multiplicative noise, that for a system of constant size, a scaling exponent $\beta \neq 0$ implies that the size distribution decays faster than power law. The result holds independently of the growth rate probability density and for smaller values of $\beta \neq 0$ in the range of empirical findings.

The second main result consists of the explanation of a tent-shaped growth rate probability density as a collective phenomenon. Our model yields a growth rate probability density for firms that may be approximated by a Gaussian. Nevertheless, the aggregate growth rate pobability density of the system, for which there is empirical evidence, appears tent-shaped. This tent-shaped form is also found if firms are grouped into size bins, and a tent-shaped function is fitted to the growth rates of each bin, without the need to assume Laplacian $\mathcal{G}(g|n)$. The central idea is to take firm's size distribution into account when calculating the growth rate distribution. Small firms account for the 'fat tails', and big firms dominate in the peak of the growth rate distribution. For comparison we also show simulations of systems of Gaussian multiplicative noise with size-dependent $\sigma \propto n^{-0.25}$. Even for the latter case, which is in the range of empirical findings, the growth rate appears tent-shaped.

A scenario with more economically relevant features has been published by the authors in [17]. It may be seen as an application of the results of this paper, where firms compete in addition for purchasing power of customers. In that setting, μ is heterogeneous for firms, and can evolve over time. The entry and exit process are governed by the financial situation of companies, since they can take loans at an interest rate, and at some threshold are declared bankrupt and exit the system. These additional features modify the noise term g, but only to an extent that the results presented in this paper still hold.

Appendices

A Alternative implementation corresponding to growth of independent subunits

The following setup yields a discrete Gaussian growth rate probability density even for small firms. Firms demand on average a quantity of workers $\hat{n}_i = n_i\,(1+\mu)$, where $0 < \mu < 1$, which is not necessarily an integer number. In our simulations, instead of standard rounding, for every existing post j, the joboffer $\hat{j}$ may be 1 or 2:

$$\hat{j} = \begin{cases} 2 & \text{with probability } \mu \\ 1 & \text{with probability } 1-\mu \end{cases} \tag{13}$$

Then, the joboffer $\hat{n}_i$ is the sum of the joboffers correponding to the posts of a firm.

$$\hat{n}_i = \sum_{j=1}^{n_i} \hat{j} \tag{14}$$

This $\hat{n}_i$ is the offer posted in the job market. Then, the aggregate joboffer $\sum_i \hat{n}_i$ is collected. If available workforce N_w is inferior to this offer (which is the case studied here), every open post has a probability

$$p = \frac{N_w}{\sum_i \hat{n}_i} \approx \frac{1}{1+\mu} \tag{15}$$

of receiving a worker. This attribution on its own would yield a binomial constraint, depending on a firm's size, as equation 2 states. However if firms determine their joboffer via equations (13) and (14), the number of received workers follows a symmetric distribution between 0 and $2n$, if n was the size of the firm in the previous timestep. Combining the probabilities for a job to offer 2 posts with the constraints in the job market for each of these joboffers, a single job has a certain probability to double, and a certain probability to vanish:

$$p(j=2) = q = \frac{\mu}{(1+\mu)^2} \tag{16}$$

$$p(j=1) = p = \frac{1+\mu^2}{(1+\mu)^2} \tag{17}$$

$$p(j=0) = q = \frac{\mu}{(1+\mu)^2} \tag{18}$$

$$\tag{19}$$

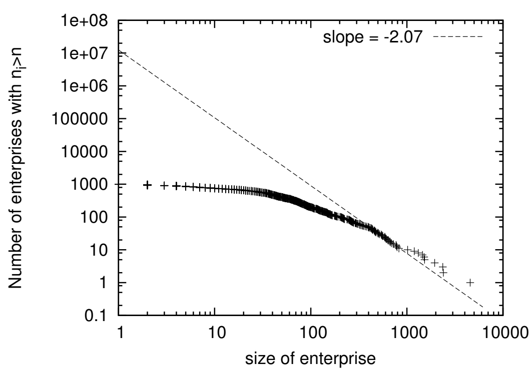
These probabilities are for single posts. For a firm of size n, the probabilities of receiving k workers can be calculated out of these probabilities p and q, in an analogous way as the coefficients of the Pascalian triangle are found. It is indeed possible to establish a recursion relation for the coefficients C. The probability that a firm of size n will have the size $k := 2n - l$ in the following timestep is given by

$$p(2n-l|n) = \sum_{j=0,n-l+2j>0,l-2j>0} \left(C(p^{l-2j-1} q^{n-l+2j}) + C(p^{l-2j} q^{n-l-2j-1}) \right) p^{l-2j}\, q^{n-l-2j} \tag{20}$$

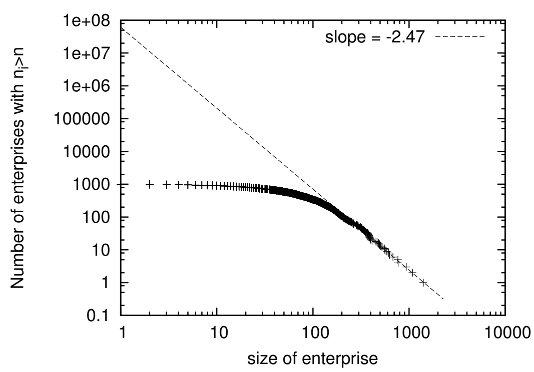
In this derivation, the re-insertion of new firms, which slightly increases the variance, has been neglected. Numerically, $G(g|n)$ is less noisy with this rounding methods, compared to the case where firms offer precisely $(1+\mu)n_i$ jobs. This rounding method is convenient because it yields a Gaussian $G(g|n)$ already for small firms, and therefore more orders of magnitude than for a can be taken into account for the confirmation of our theoretical results.

B Technical details on numerical determination of α

In simulations, we keep the condition that N_w remains constant. Whenever a firm goes bankrupt, it is replaced by a firm of size n' drawn from distribution $\mathcal{F}(n')$. The size $\sum_i n_i^{new}$ of these newly introduced firms modifies how many workers are available for existing firms, i.e. the average m around which the Gaussian $G(g)$ is centered. If for instance the new firms start with a size drawn from a distribution $\mathcal{F}(n^{new})$, with average bigger than the median of already existing firms, less workers are available for existing firms, and these will on average shrink, because the Gaussian $G(g)$ is centered around a value m slightly smaller than 1. This implies that the larger the average size at restart $\langle n^{new}\rangle$, the lower the average m, and by integrating equation (9), the steeper is the corresponding exponent α. This relationship between the additive noise f and the exponent α has been found in for a different additive noise term f by [40] and (in continuous time) by [39]: both do not allow firms to shrink below a given threshold S_{min} via an additive term. They find that the bigger S_{min}, the bigger α. Put differently, this means that α declines the bigger the average growth rate is. [39] further states that α declines the lower the death rate, and the larger the variance of the growth rate is. We reproduced these results both evaluating equation (9) as well as in numerical simulations.



(a)



(b)

Figure 8: Snapshots (after 3950 iterations) of the stationary size distribution of a system with $N_w = 10^5$ workers and $N_c = 10^3$ firms, with (size-independent) Gaussian multiplicative noise ($\sigma = 0.05$), centered around slightly different values m. The slopes are in agreement with the theoretical values: (a) $\alpha = -2.15$ (b) $\alpha = -2.45$.

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